

Bounded solutions and asymptotic stability to nonlinear second-order neutral difference equations with quasi-differences

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Abstract

This work is devoted to the study of the nonlinear second-order neutral difference equations with quasi-differences of the form

$$\Delta(r_n \Delta(x_n + q_n x_{n-\tau})) = a_n f(x_{n-\sigma}) + b_n$$

with respect to (q_n) . For $q_n \rightarrow 1$, $q_n \in (0, 1)$ the standard fixed point approach is not sufficed to get the existence of the bounded solution, so we combine this method with an approximation technique to achieve our goal. Moreover, for $p \geq 1$ and $\sup |q_n| < 2^{1-p}$ using Krasnoselskii's fixed point theorem we obtain sufficient conditions of the existence of the solution which belongs to l^p space.

Keywords nonlinear neutral difference equation, Krasnoselskii's fixed point theorem, approximation.

AMS Subject classification 39A10, 39A22.

1 Introduction

Difference equations are used in mathematical models in diverse areas such as economy, biology, computer science, see, for example [1], [7]. In the past thirty years, oscillation, nonoscillation, the asymptotic behaviour and existence of bounded solutions to many types second-order difference equation

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have been widely examined, see for example [2], [4], [6], [9], [10], [11], [13], [14], [17], [18], [19], [20], [27], [28], [29], [30], [31], [32], and references therein.

The second-order difference equation with quasi-difference of the form

$$\Delta(r_n \Delta(x_n + q_n x_{n-\tau})) = F(n, x_{n-\sigma})$$

is studied in the literature with respect to a sequence (q_n) . The fixed point theory is the standard technique to prove the existence of the bounded solution to the considered problem with constant (q_n) and (q_n) which is separated from 1. Let us present short overview of papers which deal with this problem. By using Banach's fixed point theorem, Jinfa [12] and Liu et al. [15] investigated the nonoscillatory solution to the second-order neutral delay difference equation with any constant coefficients q_n it means the equation

$$\Delta(r_n \Delta(x_n + q x_{n-\tau})) + f(n, x_{n-d_{1n}}, \dots, x_{n-d_{kn}}) = c_n.$$

In fact Liu et al. [15] proved the existence of uncountable many bounded nonoscillatory solutions for the above problem under Lipschitz continuity condition. By Leray-Schauder type of condensing operators Agarwal et al. [2] examined the existence of a nonoscillatory solution to the problem

$$\Delta(r_n \Delta(x_n + q x_{n-\tau})) + F(n+1, x_{n+1-\sigma}) = 0,$$

where $q \in \mathbb{R} \setminus \{\pm 1\}$. Liu et al. [16] discussed the existence of uncountable many bounded positive solutions to

$$\Delta(r_n \Delta(x_n + b_n x_{n-\tau} - c_n)) + f(n, x(f_1(n)), \dots, x(f_k(n))) = d_n,$$

where $\sup_{n \in \mathbb{N}} b_n = b^*$, $b^* \neq 1$ or $\inf_{n \in \mathbb{N}} b_n = b_*$, $b_* \neq -1$ by Krasnoselskii's fixed point theorem.

On the other hand, Petropoulos and Siafarikas considered different types of difference equations in the Hilbert space, see, [22], [23], [24]. Moreover, the functional-analytical method to a general nonautonomous difference equations of the form $x_{k+1} = f_k(x_k, x_{k+1})$ was considered by Pötzsche and Ey [8], Pötzsche [25]. This approach allows better characterize solutions to difference equations.

In this paper we study the following second-order neutral difference equation with quasi-difference

$$\Delta(r_n \Delta(x_n + q_n x_{n-\tau})) = a_n f(x_{n-\sigma}) + b_n, \tag{1}$$

where $\tau \in \mathbb{N} \cup \{0\}$, $\sigma \in \mathbb{Z}$, $a, b, q : \mathbb{N} \rightarrow \mathbb{R}$, $r : \mathbb{N} \rightarrow \mathbb{R} \setminus \{0\}$, and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a locally Lipschitz function. If the sequence (q_n) is convergent to 1, then fixed point approach can not be applied to solve the studied problem, because Krasnoselskii's fixed point theorem need two operators which one of them is a contraction. To overcome the limitation of this method we combine this approach with the approximation technique under the additionally assumption $q_n \in (0, 1)$. The approximation approach in this type of difference equations and the case when (q_n) is convergent to 1 has not been discussed so far, to our knowledge. Moreover, in the case $p \geq 1$ and $\sup |q_n| < 2^{1-p}$ we establish sufficient conditions of the existence the solution to (1) which belongs to l^p space. To get our result Krasnoselskii's fixed point theorem is used.

2 Preliminaries

Throughout this paper, we assume that Δ is the forward difference operator, $\mathbb{N}_k := \{k, k+1, \dots\}$ where k is a given positive integer, $\mathbb{N}_1 = \mathbb{N}$ and \mathbb{R} is a set of all real numbers.

Let $k \in \mathbb{N}$. We consider the Banach space l_k^∞ of all real bounded sequences $x : \mathbb{N}_k \rightarrow \mathbb{R}$ equipped with the standard supremum norm, i.e.

$$\|x\| = \sup_{n \in \mathbb{N}_k} |x_n|, \text{ for } x = (x_n)_{n \geq k} \in l_k^\infty.$$

Definition 1. [5] A subset A of l_k^∞ is said to be uniformly Cauchy if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}_k$ such that $|x_i - x_j| < \varepsilon$ for any $i, j \geq n_0$ and $x = (x_n) \in A$.

Theorem 1. [5] A bounded, uniformly Cauchy subset of l_k^∞ is relatively compact.

For a real $p \geq 1$ we define l^p the Banach space of p -summable sequences as follows

$$l^p := \{x : \mathbb{N} \rightarrow \mathbb{R} : \sum_{n=1}^{\infty} |x(n)|^p < \infty\}$$

with the standard norm, i.e.

$$\|x\|_{l^p} = \left(\sum_{n=1}^{\infty} |x(n)|^p \right)^{1/p}.$$

The relative compactness criterion in l^p is given in the following theorem

Theorem 2. ([3], p.106) *Let $p \in [1, \infty)$. A subset A of l^p is relatively compact if and only if A is bounded and*

$$\lim_{l \rightarrow \infty} \sup_{x=(x_n) \in A} \sum_{n=l}^{\infty} |x_n|^p = 0.$$

To get main results of this paper we use Krasnoselskii's fixed point theorem of the form.

Theorem 3. ([33], 11.B p. 501) *Let X be a Banach space, B be a bounded, closed, convex subset of X and $S, G : B \rightarrow X$ be mappings such that $Sx + Gy \in B$ for any $x, y \in B$. If S is a contraction and G is a compact, then the equation*

$$Sx + Gx = x$$

has a solution in B .

To use the approximation technique we need the following the Banach-Alaoglu theorem.

Theorem 4. [26] *If X is Banach space and $S^* = \{x^* \in X^* : \|x^*\| \leq 1\}$, then S^* is weak*-compact.*

Let us close the preliminaries paragraph by definitions of different types of solution to (1). By a solution to equation (1) we mean a sequence $x : \mathbb{N}_k \rightarrow \mathbb{R}$ which satisfies (1) for every $n \in \mathbb{N}_k$ for some $k \geq \max\{\tau, \sigma\}$. By a full solution to equation (1) we mean a sequence $x : \mathbb{N}_{\max\{\tau, \sigma\}} \rightarrow \mathbb{R}$ which satisfies (1) for every $n \geq \max\{\tau, \sigma\}$. For $p \geq 1$, a solution x to (1) is said to l^p solution, if $x \in l^p$.

3 The existence of bounded solutions respect to sequence (q_n)

In this section, sufficient conditions for the existence of a bounded solution to equation (1) respect to values of sequence (q_n) are derived.

In this section, unless otherwise note, we assume $\tau \in \mathbb{N} \cup \{0\}$, $\sigma \in \mathbb{Z}$, $a, b, q : \mathbb{N} \rightarrow \mathbb{R}$, $r : \mathbb{N} \rightarrow \mathbb{R} \setminus \{0\}$ and $f : \mathbb{R} \rightarrow \mathbb{R}$.

Theorem 5. *Assume that*

(H_{fl}) *f is a locally Lipschitz function;*

$$(H_s) \quad \sum_{s=1}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |a_t| < +\infty, \quad \sum_{s=1}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |b_t| < +\infty,$$

$$(H_q) \quad \sup_{n \in \mathbb{N}} |q_n| = q^* < 1.$$

Then, the equation (1) possesses a bounded solution.

Proof. Let $M > 0$. From the continuity of f on $[-M, M]$ we get the existence of $Q > 0$ such that

$$|f(x)| \leq Q, \text{ for } x \in [-M, M].$$

By (H_s) there exists $n_0 > \beta := \max\{\tau, \sigma\}$ such that

$$\sum_{s=n_0}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} (|a_t|Q + |b_t|) < (1 - q^*)M. \quad (2)$$

We consider the Banach space l_1^∞ and its subset

$$A_{n_0} = \{x = (x_n)_{n \in \mathbb{N}_1} \in l_1^\infty : x_1 = \dots = x_{n_0+\beta-1} = 0, |x_n| \leq M, n \geq n_0 + \beta\}.$$

Observe that A_{n_0} is a nonempty, bounded, convex and closed subset of l_1^∞ . Define two mappings $T_1, T_2: l_1^\infty \rightarrow l_1^\infty$ as follows

$$(T_1x)_n = \begin{cases} 0, & \text{for } 1 \leq n < n_0 + \beta \\ -q_n x_{n-\tau}, & \text{for } n \geq n_0 + \beta \end{cases}$$

$$(T_2x)_n = \begin{cases} 0, & \text{for } 1 \leq n < n_0 + \beta \\ \sum_{s=n}^{\infty} \frac{1}{r_s} \sum_{t=s}^{\infty} (a_t f(x_{t-\sigma}) + b_t), & \text{for } n \geq n_0 + \beta. \end{cases}$$

Our next goal is to check assumptions of Theorem 3 - Krasnoselskii's fixed point.

Firstly, we show that $T_1x + T_2y \in A_{n_0}$ for $x, y \in A_{n_0}$. Let $x, y \in A_{n_0}$. For $n < n_0 + \beta$ $(T_1x + T_2y)_n = 0$. For $n \geq n_0 + \beta$ from assumption (H_q) and (2) we get

$$|(T_1x + T_2y)_n| \leq |q_n x_{n-\tau}| + \sum_{s=n}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} (|a_t|Q + |b_t|) \leq q^*M + (1 - q^*)M = M.$$

It is easy to see that

$$\|T_1x - T_1y\| \leq q^* \|x - y\|, \text{ for } x, y \in A_{n_0},$$

so that T_1 is a contraction.

To prove the continuity of T_2 , we note assumption (H_{fl}) implies that f is Lipschitz function on $[-M, M]$, say, with constant $L > 0$, which means

$$|f(u) - f(v)| \leq L|u - v|, \text{ for } u, v \in [-M, M].$$

Hence

$$\|T_2x - T_2y\| \leq L \left(\sum_{s=1}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |a_t| \right) \|x - y\|, \text{ for } x, y \in A_{n_0}.$$

Actually, we prove that T_2 is the Lipschitz operator.

Now we show that $T_2(A_{n_0})$ is uniformly Cauchy. Let $\varepsilon > 0$. From (H_s) we get the existence of $n_\varepsilon \in \mathbb{N}$ such that

$$2 \sum_{s=n_\varepsilon}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} (|a_t|Q + |b_t|) < \varepsilon.$$

For $m > n \geq n_\varepsilon \geq n_0 + \beta$ and for $x \in A_{n_0}$ we have

$$\begin{aligned} |(T_2x)_n - (T_2x)_m| &= \left| \sum_{s=n}^{\infty} \frac{1}{r_s} \sum_{t=s}^{\infty} (a_t f(x_{t-\sigma}) + b_t) - \sum_{s=m}^{\infty} \frac{1}{r_s} \sum_{t=s}^{\infty} (a_t f(x_{t-\sigma}) + b_t) \right| \\ &\leq 2 \sum_{s=n_\varepsilon}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} (|a_t|Q + |b_t|) < \varepsilon. \end{aligned}$$

Since $T_2(A_{n_0})$ is uniformly Cauchy and bounded then by Theorem 1 $T_2(A_{n_0})$ is relatively compact in l^∞ which means that T_2 is a compact operator.

From Krasnosielskii's theorem we get that there exists $x = (x_n)_{n \in \mathbb{N}_1}$ the fixed point of $T_1 + T_2$ on A_{n_0} . Applying operator Δ to both sides of the above equation and multiplying by r_n and applying operator Δ second time for $n \geq n_0 + \beta$ we get $x = (x_n)_{n \in \mathbb{N}_{n_0+\beta}}$ is the solution to (1). \square

Corollary 1. *Assume that*

(H_{fl}) *f is a locally Lipschitz function;*

$$(H'_s) \sum_{s=\sigma+1}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=\sigma}^{s-1} |a_t| < +\infty, \quad \sum_{s=\sigma+1}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=\sigma}^{s-1} |b_t| < +\infty,$$

$$(H_q) \sup_{n \in \mathbb{N}} |q_n| = q^* < 1.$$

Then, the equation (1) possesses a bounded solution.

Proof. The proof is analogous to the proof of the Theorem 5 with

$$(T_2 x)_n = \begin{cases} 0, & \text{for } 1 \leq n < n_0 + \beta \\ - \sum_{s=n}^{\infty} \frac{1}{r_s} \sum_{t=\sigma}^{s-1} (a_t f(x_{t-\sigma}) + b_t), & \text{for } n \geq n_0 + \beta \end{cases}$$

where for any $M > 0$ there exist $Q > 0$ and $n_0 > \beta := \max\{\tau, \sigma\}$ such that

$$\sum_{s=n_0}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=\sigma}^{s-1} (|a_t|Q + |b_t|) < (1 - q^*)M.$$

□

Remark 1. Assumptions (H_s) and (H'_s) are not comparable. Indeed, let us consider sequences $(a_n), (b_n), (r_n)$, where $a_n = \frac{1}{(2n-1)(2n+1)}$, $b_n = 0$, $r_n = \sqrt{n}$ for $n \in \mathbb{N}$, (see [21]). Then

$$\sum_{s=1}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |a_t| = \sum_{s=1}^{\infty} \frac{1}{\sqrt{s}} \sum_{t=s}^{\infty} \frac{1}{(2t-1)(2t+1)} = \sum_{s=1}^{\infty} \frac{1}{2\sqrt{s}(2s-1)} < \infty,$$

so assumption (H_s) of Theorem 5 is satisfied. Assumption (H'_s) of Corollary 1 is not fulfilled because

$$\sum_{s=1}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=1}^{s-1} |a_t| = \sum_{s=1}^{\infty} \frac{1}{\sqrt{s}} \sum_{t=1}^{s-1} \frac{1}{(2t-1)(2t+1)} = \sum_{s=1}^{\infty} \frac{(s-1)}{\sqrt{s}(2s-1)} = \infty.$$

On the other hand, let us consider sequences $(a'_n), (b'_n), (r'_n)$, where $a'_n = n$, $b'_n = 0$, $r'_n = 2^n$ for $n \in \mathbb{N}$. Then

$$\sum_{s=1}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=1}^{s-1} |a_t| = \sum_{s=1}^{\infty} 2^{-s} \sum_{t=1}^{s-1} t = \sum_{s=1}^{\infty} 2^{-s-1} s(s-1) < \infty.$$

which means assumption (H'_s) of Corollary 1 is satisfied. To see that assumption (H_s) of Theorem 5 is not fulfilled notice that (H_s) implies that $\sum_{t=0}^{\infty} |a'_t| < \infty$, which give a contradiction for (a'_n) .

Corollary 2. *If in Theorem 5 and Corollary 1 we additionally assume*

(H'_0) $\tau, \sigma \in \mathbb{N} \cup \{0\}$, $\tau > \sigma$ and $q_n \neq 0$ for $n \in \mathbb{N}$.

Then, the equation (1) possesses a bounded full solution.

Proof. We find previous n_0 terms of sequence x by formula

$$x_{n-\tau} = \frac{1}{q_n} \left(-x_n + \sum_{s=n}^{\infty} \frac{1}{r_s} \sum_{t=s}^{\infty} (a_t f(x_{t-\sigma}) + b_t) \right),$$

or

$$x_{n-\tau} = \frac{1}{q_n} \left(-x_n + \sum_{s=n}^{\infty} \frac{1}{r_s} \sum_{t=\sigma}^{s-1} (a_t f(x_{t-\sigma}) + b_t) \right),$$

starting with putting $n := n_0 + 2\tau - 1$. □

Using the same technique we get the following result.

Theorem 6. *Assume that*

(H_0) $\tau, \sigma \in \mathbb{N} \cup \{0\}$, $\tau > \sigma$,

(H_{fl}) f is a locally Lipschitz function;

(H_s) $\sum_{s=1}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |a_t| < +\infty$, $\sum_{s=1}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |b_t| < +\infty$,

(H_q^1) $\inf_{n \in \mathbb{N}} q_n = q^ > 1$.*

Then, there exists a bounded full solution to (1).

Proof. The proof is similar to the proof of the Theorem 5 with operators

$$(T_1 x)_n = \begin{cases} 0, & \text{for } 1 \leq n < n_0 \\ -\frac{1}{q_{n+\tau}} x_{n+\tau}, & \text{for } n \geq n_0 \end{cases}$$

$$(T_2 x)_n = \begin{cases} 0, & \text{for } 1 \leq n < n_0 \\ \frac{1}{q_{n+\tau}} \sum_{s=n+\tau}^{\infty} \frac{1}{r_s} \sum_{t=s}^{\infty} (a_t f(x_{t-\sigma}) + b_t), & \text{for } n \geq n_0, \end{cases}$$

where for any $M > 0$ there exist $Q > 0$ and $n_0 > \beta := \max\{\tau, \sigma\}$ such that

$$\sum_{s=n_0}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} (|a_t|Q + |b_t|) < (1 - \frac{1}{q^*})M.$$

□

Now we use an approximation technique to get our main result of this section.

Theorem 7. *Assume that*

$$(H_0) \quad \tau, \sigma \in \mathbb{N} \cup \{0\}, \tau > \sigma,$$

(H_{fb}) *f is a locally Lipschitz function and f is a bounded function with constant P ;*

(H_{sb}) *there exist $C \in (0, 1)$ and increasing sequence $(w_k)_{k \in \mathbb{N}} \subset (0, 1)$ with $\sum_{k=1}^{\infty} (1 - w_k) < \infty$ such that*

$$\sum_{s=k}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} (|a_t| P + |b_t|) \in O \left((1 - w_k) (C w_k)^k \right);$$

$$(H_{q=1}) \quad q_n \in (0, 1), \quad n \in \mathbb{N}, \quad \lim_{n \rightarrow \infty} q_n = 1, \quad \inf_{n \in \mathbb{N}} q_n > 0.$$

Then, there exists a bounded full solution to equation (1).

Proof. For any $k \in \mathbb{N}$, let us consider an auxiliary problem

$$\Delta (r_n \Delta (x_n + w_k q_n x_{n-\tau})) = a_n f(x_{n-\sigma}) + b_n, \quad (3)$$

where (w_k) is the sequence satisfying (H_{sb}) . It is obvious that

$$\sup\{w_k q_n : n \in \mathbb{N}\} = w_k < 1.$$

Without loss of generality we can assume that

$$\inf\{q_n : n \in \mathbb{N}\} > C,$$

where C is the constant from assumption (H_{sb}) . By (H_{sb}) there exist $k_0 \in \mathbb{N}$, $D > 0$ such that for any $k \geq k_0$

$$\sum_{s=k}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} (|a_t| P + |b_t|) \leq D(1 - w_k) (C w_k)^k. \quad (4)$$

From Theorem 5, (3) possesses a bounded solution $x^k = (x_n^k)_{n \in \mathbb{N}_{n_k + \tau}}$ for some $n_k \in \mathbb{N}$. Moreover, by the proof of the Theorem 5 we see that (4) implies (2) with $M_k = D(C w_k)^k$. From the Theorem 5 we get $x^k = (x_n^k)_{n \in \mathbb{N}_{n_k + \tau}} \in l_{n_k + \tau}^{\infty}$

as the fixed point of the $T_1 + T_2$ on A_{n_k} . By (4) $n_k := k$ for $k \geq k_0$. It means that for $k \geq k_0$, $x^k = (x_n^k)$ solve (3) for $n \geq k + \tau$ and $|x_n^k| \leq D(Cw_k)^k$ for $n \geq k + \tau$. We find previous k terms of sequence $(x_n^k)_{n \geq \tau}$ by formula

$$x_{n-\tau}^k = \frac{1}{w_k q_n} \left(-x_n^k + \sum_{s=n}^{\infty} \frac{1}{r_s} \sum_{t=s}^{\infty} (a_t f(x_{t-\sigma}) + b_t) \right).$$

Putting $n := n_0 + 2\tau - 1 = k + 2\tau - 1$ to above we get

$$\begin{aligned} |x_{n_0+\tau-1}^k| &= |x_{k+\tau-1}^k| \leq \frac{1}{Cw_k} \left(D(Cw_k)^k + \sum_{s=k+2\tau-1}^{\infty} \frac{1}{|r_s|} \sum_{t=s}^{\infty} (|a_t|P + |b_t|) \right) \\ &\leq \frac{1}{Cw_k} \left(D(Cw_k)^k + \sum_{s=k}^{\infty} \frac{1}{|r_s|} \sum_{t=s}^{\infty} (|a_t|P + |b_t|) \right) \leq D(1 + (1 - w_k))(Cw_k)^{k-1}. \end{aligned}$$

For $k \geq 2$

$$\begin{aligned} |x_{n_0+\tau-2}^k| &= |x_{k+\tau-2}^k| \leq \frac{1}{Cw_k} \left(|x_{k+2\tau-2}| + \sum_{s=k+2\tau-2}^{\infty} \frac{1}{|r_s|} \sum_{t=s}^{\infty} (|a_t|P + |b_t|) \right) \\ &\leq \begin{cases} \frac{1}{Cw_k} \left(D(1 + (1 - w_k))(Cw_k)^{k-1} + \sum_{s=k}^{\infty} \frac{1}{|r_s|} \sum_{t=s}^{\infty} (|a_t|P + |b_t|) \right) & \text{for } \tau = 1 \\ \frac{1}{Cw_k} (D(Cw_k)^k + D(1 - w_k)(Cw_k)^k) & \text{for } \tau \geq 2 \end{cases} \\ &\leq \begin{cases} D(1 + 2(1 - w_k))(Cw_k)^{k-2} & \text{for } \tau = 1 \\ D(1 + (1 - w_k))(Cw_k)^{k-1} & \text{for } \tau \geq 2 \end{cases} \end{aligned}$$

We give the estimation of $|x_n^k|$ for the case $\tau = 1$, $\sigma = 0$. The other case are analogous and are left to the reader. Indeed,

$$\begin{aligned} |x_{k-2}^k| &\leq (Cw_k)^{-1} \left(|x_{k-1}^k| + \sum_{s=k-1}^{\infty} \frac{1}{|r_s|} \sum_{t=s}^{\infty} (|a_t|P + |b_t|) \right) \\ &\leq (Cw_k)^{k-3} (D + 2D(1 - w_k)) + D \frac{(Cw_{k-1})^{k-1}}{Cw_k} (1 - w_{k-1}) \\ &\leq (Cw_k)^{k-3} (D + 2D(1 - w_k)) + D \frac{(Cw_k)^{k-1}}{Cw_k} (1 - w_{k-1}) \\ &\leq (Cw_k)^{k-3} (D + 2D(1 - w_k) + D(1 - w_{k-1})) \leq D + 2D(1 - w_k) + D(1 - w_{k-1}). \end{aligned}$$

By induction for $i = 3, \dots, k - k_0 + 1$

$$\begin{aligned} |x_{k-i}^k| &\leq D(Cw_k)^{k-i-1} \left(1 + 2(1 - w_k) + \sum_{j=k-i+1}^{k-1} (1 - w_j) \right) \\ &\leq 2D + D \sum_{i=1}^{\infty} (1 - w_i). \end{aligned}$$

Moreover,

$$|x_{k_0-2}^k| \leq \frac{1}{Cw_1} \left(2D + D \sum_{i=1}^{\infty} (1 - w_i) + \sum_{s=k_0-1}^{\infty} \frac{1}{|r_s|} \sum_{t=s}^{\infty} (|a_t|P + |b_t|) \right)$$

and by induction

$$|x_1^k| \leq \frac{1}{Cw_1} \left(2D + D \sum_{i=1}^{\infty} (1 - w_i) + \sum_{j=2}^{k_0-1} \sum_{s=j}^{\infty} \frac{1}{|r_s|} \sum_{t=s}^{\infty} (|a_t|P + |b_t|) \right).$$

Hence, for $n \in \mathbb{N}_\tau$, $k \geq k_0$

$$|x_n^k| \leq \frac{1}{Cw_1} \left(2D + D \sum_{i=1}^{\infty} (1 - w_i) + \sum_{j=2}^{k_0-1} \sum_{s=j}^{\infty} \frac{1}{|r_s|} \sum_{t=s}^{\infty} (|a_t|P + |b_t|) \right).$$

It means that the sequence $(x^k)_{k \geq k_0}$ is bounded in l_τ^∞ . Since $(l_\tau^1)^* = l_\tau^\infty$, then from the Banach-Alaoglu theorem we get that there exists $(x^{k_l})_{l \in \mathbb{N}} \subset (x^k)_{k \in \mathbb{N}}$ which convergent on its coordinates. It means that there exists $\bar{x} = (\bar{x}_n)_{n \in \mathbb{N}_\tau} \in l_\tau^\infty$ such that

$$\lim_{l \rightarrow \infty} x_n^{k_l} = \bar{x}_n, \text{ for } n \in \mathbb{N}_\tau$$

and

$$|\bar{x}_n| \leq \frac{1}{Cw_1} \left(2D + D \sum_{i=1}^{\infty} (1 - w_i) + \sum_{j=1}^{k_0-1} \sum_{s=j}^{\infty} \frac{1}{|r_s|} \sum_{t=s}^{\infty} (|a_t|P + |b_t|) \right), \quad (5)$$

for $n \in \mathbb{N}_\tau$. To get our results we pass with $l \rightarrow \infty$ in

$$x_n^{k_l} + w_{k_l} q_n x_{n-\tau}^{k_l} = \sum_{s=n}^{\infty} \frac{1}{r_s} \sum_{t=s}^{\infty} (a_t f(x_{t-\sigma}^{k_l}) + b_t), \text{ for } n \geq \tau. \quad (6)$$

It is easy to see that for $n \geq \tau$ we have

$$\lim_{l \rightarrow \infty} \left(x_n^{k_l} + w^{k_l} q_n x_{n-\tau}^{k_l} \right) = \bar{x}_n + q_n \bar{x}_{n-\tau}.$$

From Lebesgue's dominated convergence theorem and the continuity of f we get

$$\begin{aligned} \lim_{l \rightarrow \infty} \left(\sum_{s=n}^{\infty} \frac{1}{r_s} \sum_{t=s}^{\infty} \left(a_t f(x_{t-\sigma}^{k_l}) + b_t \right) \right) &= \sum_{s=n}^{\infty} \frac{1}{r_s} \sum_{t=s}^{\infty} \left(a_t \left(\lim_{l \rightarrow \infty} f(x_{t-\sigma}^{k_l}) \right) + b_t \right) = \\ &= \sum_{s=n}^{\infty} \frac{1}{r_s} \sum_{t=s}^{\infty} (a_t f(\bar{x}_{t-\sigma}) + b_t). \end{aligned}$$

From (6) we get that

$$\bar{x}_n + q_n \bar{x}_{n-\tau} = \sum_{s=n}^{\infty} \frac{1}{r_s} \sum_{t=s}^{\infty} (a_t f(\bar{x}_{t-\sigma}) + b_t),$$

for $n \geq \tau$. Applying operator Δ to both sides of the above equation and multiplying by r_n and applying operator Δ second time we get

$$\Delta(r_n \Delta(\bar{x}_n + q_n \bar{x}_{n-\tau})) = a_n f(\bar{x}_{n-\sigma}) + b_n, \text{ for } n \geq \tau.$$

From (5) we get that $(\bar{x}_n)_{n \geq \tau}$ is the bounded full solution to (1). \square

Now, we present an example of equation which can be considered by our method.

Example 1. The following problem

$$\Delta((-1)^n \Delta(x_n + (1 - \frac{1}{2^n}) x_{n-3})) = \frac{3}{4} \frac{1}{2^n} (\sin x_{n-1})^6, \quad n \geq 3 \quad (7)$$

with $\tau = 3$, $\sigma = 1$, $r_n = (-1)^n$, $q_n = 1 - \frac{1}{2^n}$, $a_n = \frac{3}{4} \frac{1}{2^n}$, $b_n = 0$, $n \geq 1$ and an bounded $f(x) = (\sin x)^6$ fulfil assumptions of the Theorem 7. Indeed, because f is a locally Lipschitz function as $f \in C^1(\mathbb{R})$, we have to check only (H_{sb}) . For $C = 9/10$ and $w_k = 1 - (5/8)^k$, $k \geq 1$ and $P = 1$ we get for there exists k_0 such that for any $k \geq k_0$

$$3 \left(\frac{8}{9} \right)^k < \left(1 - \left(\frac{5}{8} \right)^k \right)^k.$$

So for any $k \geq k_0$

$$\sum_{s=k}^{\infty} \sum_{t=s}^{\infty} |a_t| = \sum_{s=k}^{\infty} \sum_{t=s}^{\infty} \frac{3}{4} \frac{1}{2^s} = 3 \frac{1}{2^k} < \left(\frac{9}{10}\right)^k \left(1 - \left(\frac{5}{8}\right)^k\right)^k \left(\frac{5}{8}\right)^k.$$

It is easy to see that $x_n = (-1)^n$ is the bounded solution to (7).

Using the same technique we get the following result.

Theorem 8. *Assume that*

(H_0) $\tau, \sigma \in \mathbb{N} \cup \{0\}$, $\tau > \sigma$,

(H_{fb}) f is a locally Lipschitz function and f is a bounded function with the constant P ;

(H_{sb}) there exists decreasing sequence $(w_k)_{k \in \mathbb{N}} \subset (1, \infty)$ with $\sum_{k=1}^{\infty} (w_k - 1) < \infty$ such that

$$\sum_{s=k}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |(a_t|P + |b_t|) \in O((w_k - 1) w_k^{-k});$$

$(H_{q=1}^1)$ $q_n > 1$, $n \in \mathbb{N}$, $\lim_{n \rightarrow \infty} q_n = 1$.

Then, there exists a bounded full solution to (1).

Remark 2. *Analogous theorems we get if we change the assumption $(H_{q=1})$ or $(H_{q=1}^1)$ to one of assumptions*

$(H_{q=-1})$ $q_n \in (-1, 0)$, $n \in \mathbb{N}$, $\lim_{n \rightarrow \infty} q_n = -1$, $\sup_{n \in \mathbb{N}} q_n > 0$.

$(H_{q=-1}^1)$ $q_n < -1$, $n \in \mathbb{N}$, $\lim_{n \rightarrow \infty} q_n = -1$.

4 The existence of l^p -solution

In this section, we also assume $\tau \in \mathbb{N} \cup \{0\}$, $\sigma \in \mathbb{Z}$, $a, b, q : \mathbb{N} \rightarrow \mathbb{R}$, $r : \mathbb{N} \rightarrow \mathbb{R} \setminus \{0\}$ and $f : \mathbb{R} \rightarrow \mathbb{R}$.

Theorem 9. *Assume that*

(H_p) $p \geq 1$;

(H_{fl}) f is a locally Lipschitz function;

$$(H_{sp}) \sum_{n=1}^{\infty} \left(\sum_{s=n}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |a_t| \right)^p < +\infty, \quad \sum_{n=1}^{\infty} \left(\sum_{s=n}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |b_t| \right)^p < +\infty,$$

$$(H_{qp}) \sup_{n \in \mathbb{N}} |q_n| = q^* \in (0, 2^{1-p}).$$

Then, equation (1) possesses a l^p -solution.

Proof. From the continuity of f on $[-1, 1]$ we get the existence of $W > 0$ such that

$$|f(x)| \leq W, \text{ for } x \in [-1, 1].$$

The assumption (H_{sp}) implies there exists $n_0 > \beta := \max\{\tau, \sigma\}$ such that

$$4^{p-1} \left[W^p \sum_{n=n_0}^{\infty} \left(\sum_{s=n}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |a_t| \right)^p + \sum_{n=n_0}^{\infty} \left(\sum_{s=n}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |b_t| \right)^p \right] < 1 - 2^{p-1} q^*. \quad (8)$$

We consider Banach space l^p and its subset

$$A_{n_0} = \{x = (x_n)_{n \in \mathbb{N}} \in l^p : x_1 = \dots = x_{n_0+\beta-1} = 0, \|x\|_{l^p} \leq 1\}.$$

Observe that A_{n_0} is a nonempty, bounded, convex and closed subset of l^p . Define two mappings $T_1, T_2 : l^p \rightarrow l^p$ as follows

$$(T_1 x)_n = \begin{cases} 0, & \text{for } 1 \leq n < n_0 + \beta \\ -q_n x_{n-\tau}, & \text{for } n \geq n_0 + \beta \end{cases}$$

$$(T_2 x)_n = \begin{cases} 0, & \text{for } 1 \leq n < n_0 + \beta \\ \sum_{s=n}^{\infty} \frac{1}{r_s} \sum_{t=s}^{\infty} (a_t f(x_{t-\sigma}) + b_t), & \text{for } n \geq n_0 + \beta. \end{cases}$$

Now, we prove that assumptions of Theorem 3 - Krasnoselskii's fixed point. Firstly, we show that $T_1 x + T_2 y \in A_{n_0}$ for $x, y \in A_{n_0}$. Let $x, y \in A_{n_0}$. For $n < n_0 + \beta$ $(T_1 x + T_2 y)_n = 0$. Using twice the classical inequality

$$(x + y)^p \leq 2^{p-1}(x^p + y^p) \text{ for } x, y \geq 0, p \geq 1$$

for $n \geq n_0 + \beta$ we get

$$\begin{aligned}
|(T_1x + T_2y)_n|^p &\leq 2^{p-1} \left[(q^*)^p |x_{n-\tau}|^p + \left(\sum_{s=n}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} (|a_t|W + |b_t|) \right)^p \right], \\
&\leq 2^{p-1} \left[q^* |x_{n-\tau}|^p + 2^{p-1} \left(W^p \left(\sum_{s=n}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |a_t| \right)^p + \left(\sum_{s=n}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |b_t| \right)^p \right) \right] \\
&\leq 2^{p-1} q^* |x_{n-\tau}|^p + 4^{p-1} \left[W^p \left(\sum_{s=n}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |a_t| \right)^p + \left(\sum_{s=n}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |b_t| \right)^p \right].
\end{aligned}$$

By (8) we obtain that

$$\begin{aligned}
\|T_1x + T_2y\|_{l_p}^p &\leq 2^{p-1} q^* \|x\|_{l_p}^p + 4^{p-1} \left[W^p \sum_{n=n_0+\beta}^{\infty} \left(\sum_{s=n}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |a_t| \right)^p \right. \\
&\quad \left. + \sum_{n=n_0+\beta}^{\infty} \left(\sum_{s=n}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |b_t| \right)^p \right] \leq 1.
\end{aligned}$$

It is easy to see that

$$\|T_1x - T_1y\|_{l_p} \leq q^* \|x - y\|_{l_p}, \text{ for } x, y \in A_{n_0},$$

so that T_1 is a contraction.

To prove the continuity of T_2 , we note assumption (H_{fl}) implies that f is a Lipschitz function on $[-1, 1]$ which means that exists $L > 0$ such that

$$|f(u) - f(v)| \leq L|u - v|, \text{ for } u, v \in [-1, 1].$$

Hence for $x, y \in A_{n_0}$ and $n \geq n_0 + \beta$

$$\begin{aligned}
|(T_2x - T_2y)_n|^p &\leq \left(\sum_{s=n}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |a_t| L |x_{t-\sigma} - y_{t-\sigma}| \right)^p \\
&\leq L^p \left(\sum_{s=n}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |a_t| \right)^p \|x - y\|_{l_p}^p.
\end{aligned}$$

Combine above with (H_{sp}) we get that

$$\|T_2x - T_2y\|_{l_p} \leq L \left[\sum_{n=0}^{\infty} \left(\sum_{s=n}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |a_t| \right)^p \right]^{1/p} \|x - y\|_{l_p}, \text{ for } x, y \in A_{n_0}.$$

Actually, we prove that T_2 is a Lipschitz operator.

In an analogous way we get for $x \in A_{n_0}$ and $n \geq n_0 + \beta$

$$|(T_2x)_n|^p \leq 2^{p-1} \left[W^p \left(\sum_{s=n}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |a_t| \right)^p + \left(\sum_{s=n}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |b_t| \right)^p \right]$$

and hence by (H_{sp})

$$\begin{aligned} \lim_{l \rightarrow \infty} \sup_{x \in A_{n_0}} \sum_{n=l}^{\infty} |(T_2x)_n|^p &\leq \\ \lim_{l \rightarrow \infty} \sum_{n=l}^{\infty} 2^{p-1} \left[W^p \left(\sum_{s=n}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |a_t| \right)^p + \left(\sum_{s=n}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |b_t| \right)^p \right] &= 0, \end{aligned}$$

which means that $T_2(A_{n_0})$ is relatively compact subset of l^p .

From the Theorem 3 we get that there exists $x = (x_n)_{n \in \mathbb{N}_1}$ the fixed point of $T_1 + T_2$ on A_{n_0} . Applying operator Δ to both sides of the above equation and multiplying by r_n and applying operator Δ second time for $n \geq n_0 + \beta$ we get $x = (x_n)_{n \in \mathbb{N}_{n_0+\beta}}$ is the l^p -solution to (1). \square

Remark 3. *It is worth mention that for $p = 1$ assumption (H_{sb}) implies assumption (H_{sp}) .*

Now, we present an example of equation for which our method can be applied.

Example 2. Let us consider the following problem

$$\Delta((-1)^n \Delta(x_n + q_n x_{n-3})) = 2^{-n} f(x_{n-\sigma}) + \frac{1}{n(n+1)(n+2)(n+3)}, \quad n \geq 3 \quad (9)$$

with $\tau = 3$, $\sigma = 1$, $r_n = (-1)^n$, $a_n = 2^{-n}$, $b_n = \frac{1}{n(n+1)(n+2)(n+3)}$, $n \geq 1$ and any (q_n) , $\sup |q_n| < 1$, $f \in C^1(\mathbb{R})$. Note

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{s=n}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |a_t| &= 4, \\ \sum_{n=1}^{\infty} \sum_{s=n}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=s}^{\infty} |b_t| &= \sum_{n=1}^{\infty} \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \frac{1}{t(t+1)(t+2)(t+3)} \\ &= \sum_{n=1}^{\infty} \sum_{s=n}^{\infty} \frac{1}{4s(s+1)(s+2)} = \sum_{n=1}^{\infty} \frac{1}{12n(n+1)} < \infty \end{aligned}$$

which means that assumptions of the Theorem 7 are fulfilling with $p = 1$. Hence (9) has a l^1 -solution. It is obvious that this l^1 -solution is a l^p -solution for any $p > 1$.

Corollary 3. *Assume that*

(H_p) $p \geq 1$;

(H_{fl}) f is a locally Lipschitz function;

$$(H'_{sp}) \sum_{n=1}^{\infty} \left(\sum_{s=n}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=\sigma}^{s-1} |a_t| \right)^p < +\infty, \quad \sum_{n=1}^{\infty} \left(\sum_{s=n}^{\infty} \left| \frac{1}{r_s} \right| \sum_{t=\sigma}^{s-1} |b_t| \right)^p < +\infty,$$

$$(H_{qp}) \sup_{n \in \mathbb{N}} |q_n| = q^* \in (0, 2^{1-p}).$$

Then, equation (1) possesses a bounded solution.

Corollary 4. *If in Theorem 9 or Corollary 3 we additionally assume*

$$(H'_0) \quad \tau, \sigma \in \mathbb{N} \cup \{0\}, \quad \tau > \sigma \text{ and } q_n \neq 0, \quad n \in \mathbb{N}.$$

Then, equation (1) possesses full l^p -solution.

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